2015

BOOKLET NO. TEST CODE : **UGA**

Forenoon

Questions : 30	**Time : 2 hours**

Write your Name, Registration Number, Test Centre, Test Code and the Number of this Booklet in the appropriate places on the Answersheet.

This test contains 30 questions in all. For each of the 30 questions, there are four suggested answers. Only one of the suggested answers is correct. You will have to identify the correct answer in order to get full credit for that question. Indicate your choice of the correct answer by darkening the appropriate oval ⬬, completely on the answersheet.

You will get
4 marks for each correctly answered question,
0 marks for each incorrectly answered question and
1 mark for each unattempted question.

ALL ROUGH WORK MUST BE DONE ON THIS BOOKLET ONLY.
YOU ARE NOT ALLOWED TO USE CALCULATOR.

WAIT FOR THE SIGNAL TO START.

1. Let $\mathbb{C}$ denote the set of complex numbers and $S = \{z \in \mathbb{C} \mid \overline{z} = z^2\}$, where $\overline{z}$ denotes the complex conjugate of z. Then S has:
   (A) two elements (B) three elements
   (C) four elements (D) six elements

2. The number of one-to-one functions from a set with 3 elements to a set with 6 elements is
   (A) 20 (B) 120 (C) 216 (D) 720

3. Two sides of a triangle are of length 2 cm and 3 cm. Then, the maximum possible area (in cm^2) of the triangle is:
   (A) 2 (B) 3 (C) 4 (D) 6

4. The number of factors of $2^{15} \times 3^{10} \times 5^6$ which are either perfect squares or perfect cubes (or both) is:
   (A) 252 (B) 256 (C) 260 (D) 264

5. The minimum value of the function $f(x) = x^2 + 4x + \frac{4}{x} + \frac{1}{x^2}$ where $x > 0$, is
   (A) 9.5 (B) 10 (C) 15 (D) 20

6. The minimum area of the triangle formed by any tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the coordinate axes is
   (A) ab (B) $\frac{a^2+b^2}{2}$
   (C) $\frac{(a+b)^2}{4}$ (D) $\frac{a^2+ab+b^2}{3}$

7. The angle between the hyperbolas $xy = 1$ and $x^2 - y^2 = 1$ (at their point of intersection) is
   (A) $\frac{\pi}{2}$ (B) $\frac{\pi}{3}$ (C) $\frac{\pi}{4}$ (D) $\frac{\pi}{6}$

8. The population of a city doubles in 50 years. In how many years will it triple, under the assumption that the rate of increase is proportional to the number of inhabitants?
   (A) 75 years (B) 100 years
   (C) $50 \log_2(3)$ years (D) $50 \log_e(\frac{3}{2})$ years

9. We define a set $\{f_1, f_2, \ldots, f_n\}$ of polynomials to be a linearly dependent set if there exist real numbers $c_1, c_2, \ldots, c_n$, not all zero, such that $c_1 f_1(x) + \cdots + c_n f_n(x) = 0$ for all real numbers x. Which of the following is a linearly dependent set?

(A) $\{x, x^2, x^3\}$
(B) $\{x^2 - x, 2x, x^2 + 3x\}$
(C) $\{x, 2x^3, 5x^2\}$
(D) $\{x^2 - 1, 2x + 5, x^2 + 1\}$

10. A set of numbers S is said to be multiplicatively closed if $ab \in S$ whenever both $a \in S$ and $b \in S$. Let $i = \sqrt{-1}$ and ω be a non-real cube root of unity. Let

$$S_1 = \{a + b\,i \mid a, b \text{ are integers}\} \text{ and } S_2 = \{a + b\,\omega \mid a, b \text{ are integers}\}.$$

Which one of the following statements is true?
(A) Both S_1 and S_2 are multiplicatively closed.
(B) S_1 is multiplicatively closed but S_2 is not.
(C) S_2 is multiplicatively closed but S_1 is not.
(D) Neither S_1 nor S_2 is multiplicatively closed.

11. When the product of four consecutive odd positive integers is divided by 5, the set of remainder(s) is

(A) $\{0\}$ (B) $\{0, 4\}$ (C) $\{0, 2, 4\}$ (D) $\{0, 2, 3, 4\}$

12. Consider the equation $x^2 + y^2 = 2015$ where $x \geq 0$ and $y \geq 0$. How many solutions (x, y) exist such that both x and y are non-negative integers?

(A) None
(B) Exactly one
(C) Exactly two
(D) Greater than two

13. Let P be a point on the circle $x^2 + y^2 - 9 = 0$ above the x-axis, and Q be a point on the circle $x^2 + y^2 - 20x + 96 = 0$ below the x-axis such that the line joining P and Q is tangent to both these circles. Then the length of PQ is

(A) $5\sqrt{2}$ units (B) $5\sqrt{3}$ units (C) $5\sqrt{6}$ units (D) $6\sqrt{5}$ units

14. Let $S = \{(x, y) |\ x, y \text{ are positive integers}\}$ viewed as a subset of the plane. For every point P in S, let d_P denote the sum of the distances from P to the point $(8, 0)$ and the point $(0, 12)$. The number of points P in S such that d_P is the least among all elements in the set S, is

(A) 0 (B) 3 (C) 8 (D) 1

15. Let A, B and C be the angles of a triangle. Suppose that $\tan A$ and $\tan B$ are the roots of the equation $x^2 - 8x + 5 = 0$. Then $\cos^2 C - 8 \cos C \sin C + 5 \sin^2 C$ equals

(A) -1 (B) 0 (C) 1 (D) 2

16. Let $A = \{a_1, a_2, \ldots, a_{10}\}$ and $B = \{1, 2\}$. The number of functions $f : A \to B$ for which the sum $f(a_1) + \cdots + f(a_{10})$ is an even number, is

(A) 128 (B) 256 (C) 512 (D) 768

17. Define $\operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \end{cases}$

Let $f : \mathbb{R} \to \mathbb{R}$ be the function given by $f(x) = (x+1)\operatorname{sgn}(x^2-1)$ where $\mathbb{R}$ is the set of real numbers. Then the number of discontinuities of f is:

(A) 0 (B) 1 (C) 2 (D) 3

18. Suppose X is a subset of real numbers and $f : X \to X$ is a bijection (that is, one-to-one and onto) satisfying $f(x) > x$ for all $x \in X$. Then X cannot be:

(A) the set of integers (B) the set of positive integers

(C) the set of positive real numbers (D) the set of real numbers

19. The set of real numbers x satisfying the inequality $\dfrac{4x^2}{(1 - \sqrt{1 + 2x})^2} < 2x + 9$ is:

(A) $[-\frac{1}{2}, 0) \cup (0, \frac{45}{8})$ (B) $[-\frac{1}{2}, 0) \cup (\frac{45}{8}, \infty)$

(C) $[-\frac{1}{2}, 0) \cup (0, \infty)$ (D) $(0, \frac{45}{8}) \cup (\frac{45}{8}, \infty)$

20. Let $ABCDEFGHIJ$ be a 10-digit number, where all the digits are distinct. Further, $A > B > C$, $A + B + C = 9$, $D > E > F > G$ are consecutive odd numbers and $H > I > J$ are consecutive even numbers. Then A is

(A) 8 (B) 7 (C) 6 (D) 5

21. Let $A = \{(a, b, c) : a, b, c \text{ are prime numbers}, a < b < c, a+b+c = 30\}$. The number of elements in A is

(A) 0 (B) 1 (C) 2 (D) 3

22. Let $f(x) = \begin{cases} \frac{|\sin x|}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

Then $\int_{-1}^{1} f(x)dx$ is equal to

(A) $\frac{2\pi}{3}$ (B) $\frac{3\pi}{8}$ (C) $-\frac{\pi}{4}$ (D) 0

23. Let $f : (0, 2) \cup (4, 6) \to \mathbb{R}$ be a differentiable function. Suppose also that $f'(x) = 1$ for all $x \in (0, 2) \cup (4, 6)$. Which of the following is ALWAYS true?

(A) f is increasing

(B) f is one-to-one

(C) $f(x) = x$ for all $x \in (0, 2) \cup (4, 6)$

(D) $f(5.5) - f(4.5) = f(1.5) - f(0.5)$

24. Consider 50 evenly placed points on a circle with centre at the origin and radius R such that the arc length between any two consecutive points is the same. The complex numbers represented by these points form

(A) an arithmetic progression with common difference $\left(\cos(\frac{2\pi}{50}) + i\sin(\frac{2\pi}{50})\right)$

(B) an arithmetic progression with common difference $\left(R\cos(\frac{2\pi}{50}) + iR\sin(\frac{2\pi}{50})\right)$

(C) a geometric progression with common ratio $\left(\cos(\frac{2\pi}{50}) + i\sin(\frac{2\pi}{50})\right)$

(D) a geometric progression with common ratio $\left(R\cos(\frac{2\pi}{50}) + iR\sin(\frac{2\pi}{50})\right)$

25. Given two complex numbers z, w with unit modulus (i.e., $|z| = |w| = 1$), which of the following statements will ALWAYS be correct?
(A) $|z+w| < \sqrt{2}$ and $|z-w| < \sqrt{2}$
(B) $|z+w| \leq \sqrt{2}$ and $|z-w| \geq \sqrt{2}$
(C) $|z+w| \geq \sqrt{2}$ or $|z-w| \geq \sqrt{2}$
(D) $|z+w| < \sqrt{2}$ or $|z-w| < \sqrt{2}$

26. The number of points in the region $\{(x, y) : x^2 + y^2 \leq 4\}$ satisfying $\tan^4 x + \cot^4 x + 1 = 3 \sin^2 y$ is
(A) 1 (B) 2 (C) 3 (D) 4

27. If all the roots of the equation $x^4 - 8x^3 + ax^2 + bx + 16 = 0$ are positive, then $a + b$
(A) must be -8
(B) can be any number strictly between -16 and -8
(C) must be -16
(D) can be any number strictly between -8 and 0

28. Let O denote the origin and A, B denote respectively the points $(-10, 0)$ and $(7, 0)$ on the x-axis. For how many points P on the y-axis will the lengths of all the line segments PA, PO and PB be positive integers?
(A) 0 (B) 2 (C) 4 (D) infinite

29. Let $G(x) = \int_{-x^3}^{x^3} f(t)dt$, where x is any real number and f is a continuous function such that $f(t) > 1$ for all real t. Then,
(A) $G'(0) = 0$ and G has a local maximum or minimum at $x = 0$.
(B) For any real number c, the equation $G(x) = c$ has a unique solution.
(C) There exists a real number c such that $G(x) = c$ has no solution.
(D) There exists a real number c such that $G(x) = c$ has more than one solution.

30. There are $2n+1$ real numbers having the property that the sum of any n of them is less than the sum of the remaining $n+1$. Then,

(A) all the numbers must be positive

(B) all the numbers must be negative

(C) all the numbers must be equal

(D) such a system of real numbers cannot exist